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Abundant solitary wave structures of the nonlinear coupled scalar field equations

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Abstract. It is shown that there may be more abundant solitary wave structures of the nonlinear coupled scalar field than those of single scalar fields. In this paper, starting from a known simple example which is used in particle physics and condensed matter physics, we obtained various exact solitary wave and conoidal wave solutions by solving ϕ^4, ϕ^3, $\phi + \phi^3$, $\phi^3 + \phi^4$, ϕ^6, ϕ^5 and ϕ^α models. Generally, from an arbitrary given single scalar field we may obtain a subset of solutions which are also solutions of the nonlinear coupled scalar fields.

1. Introduction

To describe the complicated physics phenomena, physicists and mathematicians have established various nonlinear models. Usually, one has to use different methods to find some exact solutions for different models. It is interesting that if we can find some useful solutions of a model from other models. In some special cases, one may establish some completely equivalent relations among models that are used in quite different categories. For instance, the well known sine–Gordon (sG) system in (1+1)-dimensions is equivalent to the massive Thirring model [1], to the two-dimensional coulomb gas [2], to the continuous limit of the lattice x–y–z spin-$\frac{1}{2}$ model [3] and to the massive O(2) nonlinear σ model [4]. In some other cases, though the model is not completely equivalent, there may still be some relations among their special solutions. In [5], we have mapped the special solutions of the constrained cubic nonlinear Klein–Gordon (^{3}NKG or ϕ^4) equation to those of the sG, double sG (DsG) , Ginzburg–Landau (GL), Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations. In [6], some special solutions of the simple models (sG and ϕ^4) have been deformed to those of the complex models (DsG, ϕ^6 and $\phi^4 + \phi^3$).

In this paper we study the solitary wave structure of the nonlinear coupled scalar fields (NCSF) ψ and ϕ which satisfy

$$\Box\psi \equiv \sum_{i=1}^{D} \psi_{x_i x_i} - \psi_{tt} = a_1\psi + a_2\psi^3 + a_3\phi^2\psi \tag{1}$$

$$\Box\phi = b_1\phi + b_2\phi^3 + b_3\psi^2\phi \tag{2}$$

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by using some single nonlinear Klein–Gordon (NKG) fields. The lower-dimensional form of the system (1) and (2) appears in some different physical fields such as particle physics and field theory [7] and condensed matter physics [8].

In the one-dimensional case ($\psi_y = \psi_z = \psi_t = \phi_z = \phi_y = \phi_t = 0$) (or for travelling wave solutions of (1) and (2)), Rajaraman [7] and Wang [9] constructed three types of solitary wave solutions of the system (1) and (2) for some special constant parameters a_i, b_i. In sections 2–4, we will see that the solitary wave solutions of the NCSF are much more abundant than the known ones.

In section 2, we give a general relation among some special solutions of an arbitrary NKG field and those of the NCSF system (1) and (2). In section 3, we list some possible polynomial nonlinearities for the field ϕ and special parameters a_i, b_i. The exact solitary wave and conoidal wave solutions of the models listed in section 3 are discussed in section 4. Section 5 is a short summary and discussion.

2. Special solutions of NCSF from a single Klein–Gordon field

Notice that the systems (1) and (2) are form invariant under the transformations $\psi \to \pm\psi$ and $\phi \to \pm\phi$, we can write ψ as

$$\psi = \sqrt{(\Box\phi - (b_1\phi + b_2\phi^3))/(b_3\phi)} \tag{3}$$

from (2). Substituting (3) into (1) we have ($x_0 \equiv \mathrm{i}t$)

$$\begin{aligned}\sum_{i=0}^{D}\Bigg\{&\frac{-b_3}{4}(\phi\Box\phi_{x_i} - \phi_{x_i}\Box\phi - 2b_2\phi^3\phi_{x_i})^2\\ &+\frac{b_3}{2}(2\phi_{x_i}^2\Box\phi - \phi(\Box\phi)(\Box\phi_{x_i}) - 2\phi\phi_{x_i}\Box\phi_{x_i} + \phi^2\Box\phi_{x_ix_i}\\ &+(b_1\phi^2 + b_2\phi^4)\Box\phi_{x_i} + 2b_1\phi_{x_i}^2 - (b_1 + 3b_2\phi^2)\phi^2\phi_{x_ix_i})(\Box\phi - (b_1\phi + b_2\phi^3))\Bigg\}\\ &-(a_1\phi^2b_3 + a_2\phi(\Box\phi - (b_1\phi + b_2\phi^3)) + a_3b_3\phi^4)(\Box\phi - (b_1\phi + b_2\phi^3))^2 = 0.\end{aligned} \tag{4}$$

Using the computer algebra, say, `Maple` or `Mathematica`, one can easily prove that some special types of solutions of (4) can be solved by means of the following pair system:

$$\Box\phi = G(\phi) \tag{5}$$

$$(\tilde\nabla\phi)^2 \equiv \sum_{i=1}^{D}\phi_{x_i}^2 - \phi_t^2 = F(\phi) \tag{6}$$

with $G(\phi) \equiv G$ being an arbitrary function of ϕ and $F(\phi) \equiv F$ being given by

$$\begin{aligned}F = \frac{-2\phi}{b_3W}&((2a_2 + b_3)G^3 + (b_3(2a_1\phi + \phi^3b_2 - \phi b_1 + 2a_3\phi^3) - 6a_2(\phi b_1 + \phi^3b_2))G^2\\ &+(6a_2\phi^2(\phi^2b_2 + \phi b_1)^2 - b_3(4a_3\phi^4b_1 + 4a_3\phi^6b_2\\ &+2\phi^6b_2^2 + 4a_1\phi^2b_1 + 4a_1\phi^4b_2))G\\ &+b_3(4a_3\phi^7b_1b_2 + 4a_1\phi^5b_1b_2 + 2a_1\phi^3b_1^2\\ &+2a_1\phi^7b_2^2 - 2\phi^4b_2b_1G + 2a_3\phi^9b_2^2 + 2a_3\phi^5b_1^2)\\ &-a_2(6\phi^7b_1b_2^2 + 6\phi^5b_1^2b_2 + 2\phi^3b_1^3 + 2\phi^9b_2^3) + b_3(\phi^4b_2 + \phi^2b_1 - \phi G)GG_\phi)\end{aligned} \tag{7}$$

where

$$W = 4G\phi(b_1 + 3b_2\phi^2) - 4\phi^4 b_2 b_1 - 3G^2 + 2\phi^2(\phi^3 b_2 + \phi b_1 - G)G_{\phi\phi}$$
$$-\phi(4\phi b_1 + 8\phi^3 b_2 - \phi G_\phi - 2G)G_\phi. \tag{8}$$

It is interesting that the arbitrariness of G in (5) means that for an *arbitrary* given NKGF, there may exist some special solutions which are also the solutions of the NCSF system (1) and (2). So we can conclude that the solitary wave structure of the NCSF may be quite rich.

In principle, whence a solution of the single NKGF (5) with the constraint (6) is given, the corresponding solution of the NCSF is obtained at the same time. However, to solve the general NKG equation (5) with the constraint condition (6) is still very difficult. In the next section, we restrict F and G as some special polynomials of ϕ:

$$F = F_0 + \sum_{n=1}^{N} \frac{2}{n} F_n \phi^n \qquad G = \tfrac{1}{2} F_\phi = \sum_{n=1}^{N} F_n \phi^{n-1}. \tag{9}$$

The above restriction of F and G may lead to some constraints on the parameters $\{a_i, b_i\}$ at the same time.

3. Possible polynomial solutions of *F*

For simplicity, we give out only the results for $N \leqslant 6$ and nonzero $\{a_i, b_i\}$ in this section. Substituting (5) and (6) with (9) and $N = 6$ into (4) and vanishing the coefficients of ϕ^k for different k, we have 17 overdetermined complicated algebraic equations for 13 parameters $\{F_k, (k = 0, 1, \ldots, 6), a_i, b_i, (i = 1, 2, 3)\}$. Because of the complexity of these equations we write down only their final nontrivial solutions.

Case A. Without any constraints on the model parameters

If we do not put any constraints on the model parameters $\{a_i,\ b_i\}$, we find only two possible polynomial selections of F for $N \leqslant 6$.

Case A.1.

$$F = F_0 + b_1\phi^2 + \tfrac{1}{2} b_2 \phi^4 \tag{10}$$

where F_0 is an arbitrary constant. This simple situation corresponds ψ is a trivial solution, i.e. $\psi = 0$.

Case A.2.

$$F = \frac{A}{B} - \frac{(2a_3a_1 - a_3b_1 - 2a_1b_2)b_3 - a_2b_1(2a_3 + 3b_2)}{(2a_2 - b_3)a_3 - (3a_2 - 2b_3)b_2}\phi^2 - \frac{1}{2}\frac{-a_3b_3 + a_2b_2}{-a_2 + b_3}\phi^4 \tag{11}$$

where A and B are related to $\{a_i, b_i\}$ by

$$A = 2(a_2 - b_3)(b_1 - a_1)(2a_2b_1(b_2 - a_3) + (b_2a_2 - 2b_3b_2 + a_3b_3)a_1)$$
$$B = (a_3(b_3 - 2a_2) + b_2(3a_2 - 2b_3))^2.$$

In this situation, the related scalar field ψ is determined by

$$\psi = \sqrt{\frac{b_2 - a_3}{b_3 - a_2}\phi^2 - \frac{2(b_1 - a_1)(b_2 - a_3)}{(2a_2 - b_3)a_3 - (3a_2 - 2b_3)b_2}}. \tag{12}$$

The model (5) with (9) is called the ϕ^4 model if $N = 4$ and $F_1 = F_3 = 0$. The first two subcases A.1 and A.2 are just related to the known ϕ^4 model.

Case B. $b_1 = a_1$

If we put a constraint $b_1 = a_1$ to the model equations (1) and (2), we find a further possible ϕ^4 selection with

$$F = F_0 + a_1\phi^2 - \frac{1}{2}\frac{-a_3b_3 + a_2b_2}{-a_2 + b_3}\phi^4 \tag{13}$$

where F_0 being an arbitrary constant. In this subcase, the corresponding ψ is given by

$$\psi = \sqrt{\frac{b_2 - a_3}{b_3 - a_2}}\phi. \tag{14}$$

The sech-type solitary wave solution related to this case for travelling wave and $b_1 > 0,\ b_2 < 0$ has been given in [9]. If there is no further constraint on the model parameters, we find that there is no other polynomial F selection for $N \leqslant 6$ in $b_1 = a_1$ case.

Case C. $8b_3 = 3a_2$

If the model parameter b_3 is related to a_2 by $8b_3 = 3a_2$ and $N \leqslant 6$, we have only three possible polynomial F selections as follows.

Case C.1.

$$F = \frac{1}{6}\frac{(b_1 - a_1)(4b_1 - a_1)}{3b_2 - a_3} + \left(-\frac{1}{2}a_1 + \frac{3}{2}b_1\right)\phi^2 + \left(2b_2 - \frac{1}{2}a_3\right)\phi^4 + \frac{1}{6}\frac{(18b_2^2 - 9a_3b_2 + a_3^2)}{4b_1 - a_1}\phi^6. \tag{15}$$

In this case, the related ψ field is given by

$$\psi = \sqrt{\frac{1}{2b_3}\left(\frac{(6b_2 - a_3)(3b_2 - a_3)}{(4b_1 - a_1)}\phi^4 + 2(3b_2 - a_3)\phi^2 + b_1 - a_1\right)}. \tag{16}$$

Case C.2.

$$F = -\frac{2}{3}\frac{(b_1 - a_1)(-a_1 + 4b_1)}{6b_2 - a_3} + b_1\phi^2 + \left(-\frac{1}{2}a_3 + 2b_2\right)\phi^4 + \frac{1}{6}\frac{(18b_2^2 - 9a_3b_2 + a_3^2)}{4b_1 - a_1}\phi^6. \tag{17}$$

In this subcase, ψ is given by

$$\psi = \sqrt{\frac{3b_2 - a_3}{b_3}\left(\frac{6b_2 - a_3}{4b_1 - a_1}\phi^4 + \phi^2\right)}. \tag{18}$$

Case C.3.

$$F = \frac{1}{6}\frac{(a_1 - 4b_1)(6a_1b_2 + a_3a_1 - 16a_3b_1 + 48b_1b_2)}{(6b_2 - a_3)^2} + \frac{1}{2}\frac{a_3a_1 - 6a_3b_1 - 3a_1b_2 + 24b_1b_2}{6b_2 - a_3}\phi^2 + \left(2b_2 - \frac{1}{2}a_3\right)\phi^4 - \frac{(18b_2^2 - 9a_3b_2 + a_3^2)}{6(a_1 - 4b_1)}\phi^6 \tag{19}$$

while the corresponding ψ is given by

$$\psi = \sqrt{\frac{3b_2 - a_3}{2b_3(6b_2 - a_3)(4b_1 - a_1)}((a_1 - 4b_1) + (a_3 - 6b_2)\phi^2)}. \tag{20}$$

The subcases C.1–C.3 are related to the so-called ϕ^6 model (the model (5) with $F_1 = F_3 = F_5 = 0$).

Case D. $3b_2 = 8a_3$

If $b_2 = 8a_3/3$, some special solutions of the system (1) and (2) can be obtained from the so-called $\phi^3 + \phi^4$ model ($N = 4$, $F_1 = 0$) or FL model because Friedberg and Lee use the nontopological soliton of the model to describe the confinement of the quarks [10]:

$$F_\pm = -2(b_1^2 a_2 - a_1 b_3 b_1 - 2b_1 a_2 a_1 + 4b_3 a_1^2 - 8a_1^2 a_2)\frac{-3a_2 + b_3}{a_3(b_3 - 6a_2)^2} - 2\frac{-2a_1 b_3 + 3a_2 b_1}{b_3 - 6a_2}\phi^2 \pm \frac{2}{3}\sqrt{\frac{2a_3(4a_1 - b_1)}{18a_2^2 - 9a_2 b_3 + b_3^2}}b_3\phi^3 + \frac{4}{3}a_3\phi^4 \tag{21}$$

while the ψ function is given by

$$\psi = \sqrt{\frac{\pm 2a_3(4a_1 - b_1)}{(b_3 - 3a_2)(b_3 - 6a_2)}\phi - \frac{b_1 - 4a_1}{b_3 - 6a_2}}. \tag{22}$$

Under the constraint $3b_2 = 8a_3$, if there is no further constraint on the model parameters, (21) is an only further possible polynomial selection in addition to the general cases A.1 and A.2 for $N \leqslant 6$.

Case E. $3b_2 = 8a_3$, $b_3 = a_2$

If the model parameters are restricted by $3b_2 = 8a_3$ and $b_3 = a_2$, one can change some special solutions of two types of the shifted FL model (we call (5) with (9) the shifted FL model if $N = 4$, $F_0 = 0$).

Case E.1.

$$F = \frac{6F_3}{5a_3}(a_1 + b_1)\phi + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 + \frac{2}{3}F_3\phi^3 + \frac{4}{3}a_3\phi^4 \qquad \left(F_3 = \pm\sqrt{\frac{a_3(4a_1 - b_1)}{5}}\right) \tag{23}$$

while

$$\psi^2 = \frac{1}{15b_3}(5(8a_3 - 3b_2)\phi^2 + 15F_3\phi - 3(4a_1 - b_1) + 9F_3 a_3^{-1}(a_1 + b_1)\phi^{-1}). \tag{24}$$

Case E.2.

$$F = \frac{F_3}{10a_3}(4a_1 - b_1)\phi + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 + \frac{2}{3}F_3\phi^3 + \frac{4}{3}a_3\phi^4 \qquad \left(F_3 = \pm\sqrt{\frac{a_3(4a_1 - b_1)}{5}}\right) \tag{25}$$

while

$$\psi^2 = \frac{1}{60b_3}(20(8a_3 - 3b_2)\phi^2 + 60F_3\phi - 12(4a_1 - b_1) + 3F_3 a_3^{-1}(4a_1 - b_1)\phi^{-1}). \tag{26}$$

Case F. $b_2 = a_3$, $b_3 = a_2$

Whence the model parameters b_2 and b_3 are related to a_2 and a_3 by $b_2 = a_3$ and $b_3 = a_2$, we have three possible polynomial selection of F for $N \leqslant 6$.

Case F.1. In the first subcase, the field ϕ is related to the ϕ^3 model,

$$F_\pm = \frac{2}{75a_3}(b_1^2 - 8a_1b_1 + 16a_1^2) + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 \pm \frac{2}{3}\sqrt{2a_3(a_1 - b_1)}\phi^3 \tag{27}$$

and the corresponding ψ is given by

$$\psi = \sqrt{-\frac{1}{5b_3}\left(5a_3\phi^2 \mp 5\sqrt{2a_3(a_1 - b_1)}\phi - b_1 + 4a_1\right)}. \tag{28}$$

Case F.2. The second subcase is related to the $\phi + \phi^3$ model,

$$F_\pm = \left(-\frac{88}{405}b_1^2 - \frac{8}{81}a_1b_1 + \frac{128}{405}a_1^2 \pm \frac{8\sqrt{10}}{2025}\sqrt{(b_1 - a_1)(7b_1 + 2a_1)^3}\right)\frac{\phi}{\sqrt{2a_3(a_1 - b_1)}} + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 + \sqrt{2a_3(a_1 - b_1)}\phi^3. \tag{29}$$

In this case, the field ψ is determined by

$$\psi^2 = \frac{1}{b_3}\bigg(- b_2\phi^2 + \frac{3}{2}\sqrt{2a_3(a_1 - b_1)}\phi + \frac{1}{5}(b_1 - 4a_1) - \frac{2}{2025\sqrt{a_3}}\left(5\sqrt{2(a_1 - b_1)}(11b_1 + 16a_1) \pm 2\sqrt{5(7b_1 + 2a_1)^3}\right)\frac{1}{\phi}\bigg). \tag{30}$$

Case F.3. The third subcase is solved by the special shifted FL model,

$$F_\pm = \pm\frac{4}{15}\sqrt{\frac{2a_1 - 3b_1}{5a_3}}(b_1 - 4a_1)\phi + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 + \frac{5}{3}a_3\frac{b_1 - a_1}{3b_1 - 2a_1}\phi^4 \tag{31}$$

while the field ψ is determined by

$$\psi^2 = \frac{b_1 - 4a_1}{a_2}\left(\frac{a_3}{3a_2(3b_1 - 2a_1)}\phi^2 + \frac{1}{5} \pm \frac{2}{75}\sqrt{\frac{5(2a_1 - 3b_1)}{a_3}}\frac{1}{\phi}\right). \tag{32}$$

Case G. $b_2 = a_3$, $b_3 = a_2$, $b_1 = -a_1$

If we put a further constraint $b_1 = -a_1$ on the case F, we can get three further possibilities in addition to the subcases F.1–F.3.

Case G.1. The first subcase is related to the ϕ^3 model

$$F_\pm = F_0 - 2a_1\phi^2 \pm \tfrac{4}{3}\sqrt{a_3a_1}\phi^3 \tag{33}$$

with F_0 being an arbitrary constant and

$$\psi = \sqrt{\frac{1}{b_3}\left(b_2\phi^2 \mp 2\sqrt{a_1a_2}\phi - b_1\right)}. \tag{34}$$

Case G.2. The second subcase can be described by the ϕ^4 model with

$$F = 4a_1^2F_4^{-2}(4a_3 - 3F_4) + a_1F_4^{-1}(4a_3 - 5F_4)\phi^2 + \tfrac{1}{2}F_4\phi^4 \tag{35}$$

where F_4 is an arbitrary constant. In this subcase, the field ψ is related to ϕ by

$$\psi = \sqrt{\frac{(a_3 - F_4)(4a_1 - F_4\phi^2)}{a_2F_4}}. \tag{36}$$

Case G.3. The third subcase can be also casted to the special shifted FL model with

$$F = 2F_1\phi - 2a_1\phi^2 + \tfrac{2}{3}a_3\phi^4 \tag{37}$$

where F_1 is arbitrary. The field ψ in this subcase is given by

$$\psi = \sqrt{\frac{1}{a_2}\left(\frac{1}{3}a_3\phi^2 - a_1 + F_1\frac{1}{\phi}\right)}. \tag{38}$$

Case H. $b_2 = a_3\frac{b_3-2a_2}{2b_3-3a_2}$

If the model parameters have a constraint condition

$$b_2 = a_3\frac{b_3 - 2a_2}{2b_3 - 3a_2} \tag{39}$$

some special solutions of the coupled scalar field system (1) and (2) can be obtained form the FL model with

$$F_\pm = \frac{1}{3}(9a_2^2 - 9a_2b_3 + 2b_3^2)\frac{(b_1 - 4a_1)^2}{(b_3 - 6a_2)^2a_3} - 2\frac{3a_2b_1 - 2a_1b_3}{b_3 - 6a_2}\phi^2 \\ \pm\frac{4}{3}\sqrt{\frac{a_3(a_1 - b_1)}{9a_2^2 - 9a_2b_3 + 2b_3^2}}b_3\phi^3 + a_3\frac{-a_2 + b_3}{-3a_2 + 2b_3}\phi^4 \tag{40}$$

and the ψ equation now has the form

$$\psi = \sqrt{\frac{a_3}{2b_3 - 3a_2}\phi^2 \pm 2\sqrt{\frac{a_3(a_1 - b_1)}{(b_3 - 3a_2)(2b_3 - 3a_2)}}\phi - \frac{b_1 - 4a_1}{b_3 - 6a_2}}. \tag{41}$$

Case I. $b_2 = a_3\frac{b_3-2a_2}{2b_3-3a_2}$, $b_1 = 4a_1$

Whence two constraints, (31) and $b_1 = 4a_1$, are added into (1) and (2), we get a special FL model (FL model with $F_0 = 0$) to solve the NCSF with

$$F_\pm = \frac{a_3}{3a_2 - 2b_3}\phi^2 \pm \sqrt{\frac{3a_1a_3}{(b_3 - 3a_2)(3a_2 - 2b_3)}}\phi^3 + \frac{a_3(a_2 - b_3)}{3a_2 - 2b_3}\phi^4. \tag{42}$$

The corresponding solution of ψ is related to ϕ by

$$\psi^2 = \frac{a_3}{3a_2 - 2b_3}\phi^2 \pm \frac{2\sqrt{a_3a_1}}{\sqrt{(b_3 - 3a_2)(3a_2 - 2b_3)}}\phi. \tag{43}$$

Case J. $b_2 = a_3\frac{b_3-2a_2}{2b_3-3a_2}$, $b_3 = \frac{3}{2}\frac{b_1a_2}{a_1}$

If a further constraint

$$b_3 = \frac{3}{2}\frac{b_1a_2}{a_1} \tag{44}$$

in addition to (39) is added to the system (1) and (2), a critical FL model (FL model with $F_2 = 0$) is a possible polynomial selection of F for $N \leqslant 6$:

$$F_\pm = \frac{2(a_1 - b_1)(2a_1 - b_1)}{3a_3} \pm \frac{2}{3}\frac{\sqrt{2a_3}b_1}{\sqrt{2a_1 - b_1}}\phi^3 + \frac{1}{6}\frac{a_3(2a_1 - 3b_1)}{a_1 - b_1}\phi^4. \tag{45}$$

The corresponding solution of ψ has the form

$$\psi^2 = -\frac{1}{3}\frac{a_3 a_1}{a_2(a_1 - b_1)}\phi^2 \pm \frac{2}{3}\frac{\sqrt{2a_3}a_1}{a_2\sqrt{2a_1 - b_1}}\phi - \frac{2}{3}\frac{a_1}{a_2}. \tag{46}$$

For the critical FL model there are no solitary wave solutions and this situation corresponds to the phase transition point of quark deconfinment in the FL field theory.

Case K. $b_2 = \frac{4a_3}{3}$, $b_3 = \frac{6a_2}{5}$

Under the constraints,

$$b_2 = \frac{4a_3}{3} \qquad b_3 = \frac{6a_2}{5} \tag{47}$$

the ϕ equation is related to the FL model by

$$F_\pm = \frac{1}{64a_3}(4a_1 - b_1)^2 + \left(\frac{5}{4}b_1 - a_1\right)\phi^2 \pm \frac{8}{9}\sqrt{3a_3(a_1 - b_1)}\phi^3 - \frac{1}{3}a_3\phi^4 \tag{48}$$

and the ψ equation has the form

$$\psi^2 = -\frac{5}{3}\frac{a_3}{a_2}\phi^2 \pm \frac{10}{9a_2}\sqrt{3a_3(a_1 - b_1)}\phi - \frac{5}{24a_2}(4a_1 - b_1). \tag{49}$$

Case L. $b_2 = \frac{4a_3}{3}$, $b_3 = \frac{6a_2}{5}$, $b_1 = -\frac{4}{5}a_1$

In this case, ϕ is also determined by the FL model with

$$F_\pm = F_0 - 2a_1\phi^2 \pm \frac{8}{\sqrt{15}}\sqrt{a_1 a_3}\phi^3 - \frac{1}{3}a_3\phi^4 \tag{50}$$

where F_0 is an arbitrary constant. The ψ field is given by

$$\psi^2 = -\frac{5}{3}\frac{a_3}{a_2}\phi^2 \pm \frac{2}{3a_2}\sqrt{15a_1 a_3}\phi - \frac{a_1}{a_2}. \tag{51}$$

Case M. $b_2 = \frac{a_3 b_1}{a_1 + 2b_1}$, $b_3 = \frac{a_2}{a_1}(b_1 + 2a_1)$

If the model parameters are restricted by

$$b_2 = \frac{a_3 b_1}{a_1 + 2b_1} \qquad b_3 = \frac{a_2}{a_1}(b_1 + 2a_1) \tag{52}$$

we have three possible independent F polynomial selections for $N \leqslant 6$.

Case M.1. The first case is related to the FL model,

$$F_\pm = F_0 - 2a_1\phi^2 \pm 4\sqrt{\frac{-a_3}{9(a_1 + 2b_1)}}(b_1 + 2a_1)\phi^3 + a_3\frac{b_1 + a_1}{a_1 + 2b_1}\phi^4 \tag{53}$$

where F_0 is an arbitrary constant. In this case, the field ψ is

$$\psi = \sqrt{\frac{b_1 + 2a_1}{b_3}\left(\frac{a_3}{2b_1 + a_1}\phi^2 \pm 2\sqrt{\frac{-a_3}{2b_1 + a_1}}\phi - 1\right)}. \tag{54}$$

Case M.2. The second subcase is related to the special FL model,

$$F_{\pm} = -2a_1\phi^2 \pm \frac{4}{3}\sqrt{\frac{a_3a_1}{a_2(3a_2-2b_3)}}b_3\phi^3 + a_3\frac{b_3-a_2}{2b_3-3a_2}\phi^4. \tag{55}$$

In this case, ψ has the form

$$\psi = \sqrt{\frac{a_3}{2b_3-3a_2}\phi^2 \pm 2\sqrt{\frac{a_3a_1}{a_2(3a_2-2b_3)}}\phi - \frac{a_1}{a_2}}. \tag{56}$$

Case M.3. The third subcase corresponds to the shifted FL model,

$$F = -\frac{6}{5}\frac{b_1^2-3a_1b_1-4a_1^2}{\sqrt{5(-a_3b_1+4a_3a_1)}}\phi + \left(\frac{6}{5}b_1 - \frac{4}{5}a_1\right)\phi^2 + \frac{2}{3}\sqrt{\frac{-a_3b_1+4a_3a_1}{5}}\phi^3 + \frac{4}{3}a_3\phi^4 \tag{57}$$

and

$$\psi^2 = \frac{1}{5a_2}\sqrt{5a_3(4a_1-b_1)}\left(\phi + \frac{\sqrt{b_1-4a_1}}{\sqrt{5a_3}} + \frac{3}{5a_3}(a_1+b_1)\frac{1}{\phi}\right). \tag{58}$$

Case N. $b_1 = a_1$, $b_2 = \frac{1}{3}a_3$, $b_3 = 3a_2$

In this case, the polynomial F for $N \leqslant 6$ is the special FL model,

$$F = -2a_1\phi^2 + \tfrac{2}{3}F_3\phi^3 + \tfrac{2}{3}a_3\phi^4 \tag{59}$$

where F_3 is an arbitrary constant. In this case, we have

$$\psi = \sqrt{\frac{1}{6a_2}(2a_3\phi^2 + 2F_3\phi - 6a_1)} \tag{60}$$

for ψ field.

Case O. $b_3 = \frac{a_2(3b_2-2a_3)}{2b_2-a_3}$, $b_1 = 4a_1$

In this case, a new nontrivial case for the ϕ field is given by the special FL model

$$F_{\pm} = 4a_1\phi^2 \pm 4\sqrt{\frac{a_1}{3a_3-9b_2}}(3b_2-2a_3)\phi^3 + (a_3-b_2)\phi^4. \tag{61}$$

For the ψ field, we have

$$\psi = \sqrt{\frac{2b_2-a_3}{a_2}\left(-\phi^2 \pm 2\sqrt{\frac{3a_1}{a_3-3b_2}}\phi\right)}. \tag{62}$$

Case P. $b_1 = 4a_1$, $b_2 = -8a_3$, $b_3 = a_2$

Under the constraint conditions $b_1 = 4a_1$, $b_2 = -8a_3$, $b_3 = a_2$, we get

$$F = 2F_1\phi + 4a_1\phi^2 - 4a_3\phi^4 \tag{63}$$

with arbitrary F_1. The corresponding ψ field has the form

$$\psi = \sqrt{\frac{F_1}{a_2}\frac{1}{\phi}}. \tag{64}$$

Case Q. $b_3 = \frac{5}{9}a_2$

Whence the model parameter b_3 is related to a_2 by

$$b_3 = \frac{5}{9}a_2 \tag{65}$$

we obtain some possible ϕ^5 model (the model (5) with $F = F_0 + F_2\phi^2 + \frac{2}{3}F_3\phi^3 + \frac{1}{2}F_4\phi^4 + \frac{2}{5}F_5\phi^5$) to solve the NCSF system (1) and (2).

Case Q.1. For the first subcase, the function F is given by

$$F_\pm = \frac{24(15b_2 - 8a_3)a_1^2}{25(11b_2 - 6a_3)^2} - \frac{2a_1(25b_2 - 12a_3)}{5(11b_2 - 6a_3)}\phi^2 \mp \frac{2\sqrt{a_1(5b_2 - a_3)(15b_2 - 8a_3)}}{5\sqrt{3(11b_2 - 6a_3)}}\phi^3$$
$$+(2b_2 - 4/5a_3)\phi^4 \pm \frac{1}{25\sqrt{a_1}}\sqrt{3(11b_2 - 6a_3)(5b_2 - a_3)(15b_2 - 8a_3)}\phi^5 \tag{66}$$

with a further constraint on the parameter b_1,

$$b_1 = -\frac{a_1b_2}{11b_2 - 6a_3}. \tag{67}$$

The corresponding ψ equation is

$$\psi^2 = \frac{\pm 9}{50a_2\sqrt{a_1}}\sqrt{3(6a_3 - 11b_2)(a_3 - 5b_2)(15b_2 - 8a_3)}\phi^3 + \frac{9}{25a_2}(15b_2 - 8a_3)\phi^2$$
$$-\frac{\pm 9}{25a_2}\sqrt{\frac{3a_1(5b_2 - a_3)(15b_2 - 8a_3)}{(11b_2 - 6a_3)}}\phi - \frac{27}{25}\frac{(15b_2 - 8a_3)a_1}{(11b_2 - 6a_3)a_2}. \tag{68}$$

Case Q.2. The second subcase of ϕ^5 model has the form

$$F = F_0 + F_2\phi^2 - \frac{3a_3^2}{50F_5}(5c - 1)(15c - 8)\phi^3 + \frac{4}{5}a_3(5c - 2))\phi^4 + F_5\phi^5 \tag{69}$$

if the model parameters satisfy one more restriction:

$$(15c - 8)(5c - 1)(45c - 19)$$
$$\times[b_1^2(-483\,975c^4 + 1004\,265c^3 - 727\,389c^2 + 219\,579c - 23\,616)$$
$$-8a_1b_1(11\,700c^4 + 25\,095c^3 - 42\,242c^2 + 18\,071c - 2400)$$
$$+16ca_1^2(19\,350c^3 - 24\,240c^2 + 9899c - 1325)] = 0 \tag{70}$$

in addition to the constraint (65), where $c \equiv b_2/a_3$, $A_1 \equiv -2118\,075c^4 + 2046\,375c^5 - 33\,885c^3 + 736\,239c^2 - 292\,222c + 34\,560$,

$$F_0 = -\frac{32a_1^2}{75a_3}\{[10(2025c^2 - 1940c + 432)a_1$$
$$-(54\,675c^2 - 46\,745c + 9312)b_1](15c - 8)(5c - 1)\}$$
$$\times\{2a_1A_1 + (45c - 19)(161\,325c^4 - 334\,755c^3$$
$$+242\,463c^2 - 73\,193c + 7872)b_1\}^{-1} \tag{71}$$

$$F_2 = -\frac{2}{15}\frac{10(25c - 9)(15c - 8)a_1 - (10125c^2 - 8585c + 1752)b_1}{685c^2 - 577c + 120} \tag{72}$$

and

$$F_5 = \pm\sqrt{\frac{3a_3^3}{100}\frac{(15c - 8)(5c - 1)(685c^2 - 577c + 120)}{57b_1 - 135b_1c - 20a_1 + 50ca_1}}. \tag{73}$$

The related solution of the field ψ is given by

$$\psi^2 = \frac{9F_5}{5a_2}\phi^3 + \frac{9(15c-8)a_3}{25a_2}\phi^2 - \frac{27a_3^2(15c-8)(5c-1)}{250a_2F_5}\phi - \frac{3(15c-8)(213b_1 - 665b_1c + 500ca_1 - 180a_1)}{25a_2(685c^2 - 577c + 120)}. \tag{74}$$

Case Q.3. The third subcase is

$$F = -\tfrac{2}{9}a_1\phi^2 - \tfrac{2}{5}a_3\phi^4 + \tfrac{2}{5}F_5\phi^5 \tag{75}$$

for arbitrary F_5 and

$$b_1 = \tfrac{1}{3}a_1 \qquad b_2 = \tfrac{1}{5}a_3 \tag{76}$$

while the ψ field is given by

$$\psi^2 = \frac{9}{5a_2}F_5\phi^3 - \frac{9}{5a_2}a_3\phi^2 - \frac{a_1}{a_2}. \tag{77}$$

Case Q.4. The final one has the form

$$F = \tfrac{4}{9}a_1\phi^2 + \tfrac{4}{15}a_3\phi^4 + \tfrac{2}{5}F_5\phi^5 \tag{78}$$

with the arbitrary F_5 and

$$b_1 = \tfrac{4}{9}a_1 \qquad b_2 = \tfrac{8}{15}a_3 \tag{79}$$

while

$$\psi = \sqrt{\frac{9F_5}{5a_2}\phi^3}. \tag{80}$$

Case R. ϕ^α model

To close this section, we write down two special results for *arbitrary real constant* α with $\alpha \neq 2$ and $\alpha \neq 4$.

Case R.1. If the model parameters b_i are related to a_i by

$$b_1 = \frac{4a_1}{(\alpha-2)^2} \qquad b_2 = \frac{8a_3}{\alpha(\alpha-2)} \qquad b_3 = \frac{a_2\alpha}{(\alpha-2)^2} \tag{81}$$

for arbitrary real α, one can always find a special ϕ^α model to solve the NCSF:

$$F = -\frac{2a_3}{\alpha(\alpha-4)}\phi^4 - \frac{2a_1}{(\alpha-2)^2}\phi^2 + \frac{2}{\alpha}F_\alpha\phi^\alpha \tag{82}$$

where F_α may be arbitrary. In this case, the corresponding ψ field is given by

$$\psi^2 = \frac{(\alpha-2)^2F_\alpha}{a_2\alpha}\phi^{\alpha-2} - \frac{a_3(\alpha-2)^2}{\alpha a_2(\alpha-4)}\phi^2 - \frac{a_1}{a_2}. \tag{83}$$

Case R.2. If the model parameters b_i and a_i satisfy the constraints

$$b_3 = \frac{a_2\alpha}{(\alpha-2)^2} \qquad b_1 = \frac{4a_1}{(\alpha-2)^2} \qquad b_2 = \frac{8a_3}{\alpha(\alpha-2)} \tag{84}$$

for arbitrary real α, one can find another special ϕ^α model to solve the NCSF:

$$F = \frac{4a_3}{\alpha(\alpha-2)}\phi^4 + \frac{4a_1}{(\alpha-2)^2}\phi^2 + \frac{2}{\alpha}F_\alpha\phi^\alpha \tag{85}$$

where F_α is an arbitrary constant. The corresponding ψ field is given by

$$\psi = \sqrt{\frac{F_\alpha\phi^{\alpha-2}}{a_2\alpha(\alpha-2)^2}}. \tag{86}$$

4. On the exact solutions of the NKG fields

From the last two sections, we know that whence a solution of a single NKG field is given, a corresponding solution of the NCSF system is given at the same time. So, in this section we discussed the solutions of the single NKG equations with polynomial nonlinearities.

4.1. Solutions of ϕ^4 equation

For cases A, B and G.2 of the last section, the solutions of ϕ are those of the known constrained ϕ^4 equation. In [5], we have list many exact conoidal wave solutions of the constrained ϕ^4 equation. For instance, if $F_1 < 0$, $F_4 > 0$ and write F_0 as

$$F_0 = \frac{2F_2^2k^2}{(1+k^2)^2F_4} \tag{87}$$

the ϕ^4 equation possesses the exact solution

$$\phi = \sqrt{\frac{-2k^2F_2}{(1+k^2)F_4}}\,\mathrm{sn}\left(\sqrt{\frac{-F_2}{1+k^2}}\xi\right) \tag{88}$$

where

$$\xi = \int \sqrt{B}\,\mathrm{d}g \tag{89}$$

with $B \equiv B(g)$ being an arbitrary function of g and g being any solution of the base equations

$$\Box g = \frac{1}{2}\frac{\mathrm{d}B}{\mathrm{d}g} \tag{90}$$

$$(\tilde{\nabla}g)^2 = B. \tag{91}$$

If we take

$$B = \alpha^2g^2 \tag{92}$$

we have a special solution of the base equations (90) and (91) with (92) [5],

$$\xi = \frac{1}{\alpha}\ln g \tag{93}$$

and

$$g = \left(\sum_{\gamma=1}^{N}\exp\alpha_1\theta_\gamma\right)^{\alpha_2} \tag{94}$$

with

$$\theta_\gamma = \sum_{j=1}^{D} P_\gamma^j x_j + \omega_\gamma t + \delta_\gamma \tag{95}$$

$$\sum_{j=1}^{D} (P_\gamma^j)^2 - \omega_\gamma^2 = 1 \qquad \gamma = 1, 2, \ldots, N \tag{96}$$

$$\sum_{j=1}^{D} (P_\gamma^j - P_{\gamma'}^j)^2 - (\omega_\gamma - \omega_{\gamma'})^2 = 0 \qquad (\gamma \neq \gamma', \gamma, \gamma' = 1, 2, \ldots, N) \tag{97}$$

and

$$\alpha_1^2 \alpha_2^2 = \alpha^2. \tag{98}$$

The travelling wave solution corresponds to $N = 1$ in (93) and (94) with (95) and (96). The constant k in (87) and (88) is the modulus of the Jacobi elliptic function, $\mathrm{sn}(\xi)$.

If $F_2 > 0$, $F_4 < 0$ and write F_0 as

$$F_0 = -2\frac{F_2^2 k^2 k'^2}{(k^2 - k'^2)^2 F_4} \tag{99}$$

we have

$$\phi = \sqrt{-\frac{F_2(k^2 - k'^2)}{2k^2 F_4}}\, \mathrm{cn}\left(\sqrt{\frac{F_2}{k^2 - k'^2}}\xi\right) \tag{100}$$

where $k' = \sqrt{1 - k^2}$ and ξ is same as in (89).

Taking $k \to 1$, we have two types of solitary wave solutions

$$\phi = \sqrt{\frac{-F_2}{F_4}}\, \tanh\left(\sqrt{\frac{-F_2}{2}}\xi\right) \tag{101}$$

with

$$F_0 = \frac{1}{2}\frac{F_2^2}{F_4} \tag{102}$$

and

$$\phi = \sqrt{-\frac{F_2}{2F_4}}\, \mathrm{sech}\left(\sqrt{F_2}\xi\right) \tag{103}$$

with

$$F_0 = 0 \tag{104}$$

from (88) and (100), respectively.

Two known solitary wave solutions in [7, 9] are related to (101) and (103) for case A.2, respectively, and the other known one is related to (101) by using the transformations, $\psi \leftrightarrow \phi$, $\{a_i,\ b_i\} \leftrightarrow \{b_i,\ a_i\}$.

For cases A.1, B and G.2 there are only one possible solitary wave solution for both (101) and (103). However, for case A.2, there are three nontrivial different subcases for both (101) and (103). For instance, for the condition (104), the nontrivial cases are: (i) $a_1 = b_1$, (ii) $a_1 = 2b_1$, $a_3 = 2b_2$, and (iii) $b_3 = a_2(2b_1b_2 - 2b_1a_3 + a_1b_2)/(2a_1b_2 - a_1a_3)$.

4.2. Solve the ϕ^3 model by the ϕ^4 equation

In cases F.1 and G.1 of the last section, we have to solve the constrained NKG equation

$$\Box\phi = F_2\phi + F_3\phi^2 \tag{105}$$

$$(\tilde{\nabla}\phi)^2 = F_0 + F_2\phi^2 + \tfrac{2}{3}F_3\phi^3. \tag{106}$$

It is straightforward to prove that if we make the transformation

$$\phi = g^2 + c \tag{107}$$

with c being determined by

$$\tfrac{2}{3}F_3c^3 + F_2c + F_0 = 0 \tag{108}$$

the g function satisfies the constrained ϕ^4 equation

$$\Box g = g_2g + g_4g^3 \tag{109}$$

$$(\tilde{\nabla}g)^2 = g_0 + g_2g^2 + \tfrac{1}{2}g_4g^4 \tag{110}$$

with

$$g_0 = \frac{c}{2}(F_2 + F_3c) \qquad g_2 = \tfrac{1}{4}(F_2 + 2F_3c) \qquad g_4 = \tfrac{1}{3}F_3. \tag{111}$$

All the Jacobi elliptic function solutions listed in [5] now can be used. Two standard solitary wave solutions are

$$\phi = \frac{-3(F_2 + 2F_3c)}{4F_3}\tanh^2\left(\sqrt{\frac{-(F_2 + 2F_3c)}{8}}\xi\right) + c \tag{112}$$

with

$$c(F_2 + F_3c) = \frac{3(F_2 + 2F_3c)^2}{16F_3} \tag{113}$$

and

$$\phi = -\frac{3(F_2 + 2F_3c)}{8F_3}\operatorname{sech}^2\left(\sqrt{\frac{1}{4}(F_2 + 3F_3c)}\xi\right) + c \tag{114}$$

with

$$\frac{c}{4}(2F_2 + 3F_3c) = 0. \tag{115}$$

The corresponding ψ is given by (17) with (112) and (114) for case F.1 and (20) with (112) and (114) for case G.1. It is worth pointing out that the solitary waves for ϕ are tanh or sech forms in cases A, B and G.2 while in cases F.1 and G.1 the solitary waves are in $\tanh^2$ or sech^2 forms.

4.3. Change the $\phi + \phi^3$ model to the ϕ^4 equation

For case F.2 of the last section, the related constrained NKG equation system is

$$\Box\phi = F_1 + F_2\phi + F_3\phi^2 \tag{116}$$

$$(\tilde{\nabla}\phi)^2 = 2F_1\phi + F_2\phi^2 + \tfrac{2}{3}F_3\phi^3. \tag{117}$$

It is easy to prove that if we make the transformation

$$\phi = g^2 + c \tag{118}$$

where c can be taken as any one of the following values:

$$0 \qquad \frac{3}{4F_3}\left(-F_2 \pm \sqrt{F_2^2 - \frac{16}{3}F_3F_1}\right) \tag{119}$$

then the g function satisfies the constrained ϕ^4 equations (109) and (110) with

$$g_0 = \tfrac{1}{2}(F_2c + F_3c^2 + F_1) \qquad g_2 = \tfrac{1}{4}(F_2 + 2F_3c) \qquad g_4 = \tfrac{1}{3}F_3. \tag{120}$$

4.4. *Exact solutions of the ϕ^6 model*

For case C, the constrained NKG equation system has the form

$$\Box\phi = F_2\phi + F_4\phi^3 + F_6\phi^5 \tag{121}$$

$$(\tilde{\nabla}\phi)^2 = F_0 + F_2\phi^2 + \tfrac{1}{2}F_4\phi^4 + \tfrac{1}{3}F_6\phi^6. \tag{122}$$

In [6] (Lou, Huang and Ni), we established the deformation relations between the constrained ϕ^6 system (121) and (122) and the constrained ϕ^4 system. Starting from every solution of the ϕ^4 system, we can get a corresponding solution of the ϕ^6 model. The standard pair are

$$\phi = \mathrm{sn}\left(\sqrt{\frac{12F_2C^2+9F_4C+5F_6}{6(1+k^2)C^2}}\xi\right) \times\left[C\,\mathrm{sn}^2\left(\sqrt{\frac{12F_2C^2+9F_4C+5F_6}{6(1+k^2)C^2}}\xi\right) - \frac{2C(1+k^2)(6F_4C+6F_2C^2+5F_6)}{k^2(12F_2C^2+9F_4C+5F_6)}\right]^{-1/2} \tag{123}$$

with C being determined by

$$2\left(F_2+\frac{F_4}{2C}+\frac{5F_6}{18C^2}\right)\left(F_2+\frac{F_4}{C}+\frac{5F_6}{6C^2}\right) = \frac{2k^2}{(1+k^2)^2}\left(2F_2+\frac{3F_4}{2C}+\frac{5F_6}{3C^2}\right)^2 \tag{124}$$

and

$$\phi = \mathrm{cn}\left(\sqrt{\frac{12F_2C^2+9F_4C+5F_6}{6(k'^2-k^2)C^2}}\xi\right) \times\left[C\,\mathrm{cn}^2\left(\sqrt{\frac{12F_2C^2+9F_4C+5F_6}{6(k'^2-k^2)C^2}}\xi\right) - \frac{C(k^2-k'^2)(6F_4C+6F_2C^2+5F_6)}{k^2(12F_2C^2+9F_4C+5F_6)}\right]^{-1/2} \tag{125}$$

with C being determined by

$$2\left(F_2+\frac{F_4}{2C}+\frac{5F_6}{18C^2}\right)\left(F_2+\frac{2F_4}{C}+\frac{5F_6}{2C^2}\right) = \frac{-2k^2k'^2}{(k'^2-k^2)^2}\left(2F_2+\frac{3F_4}{2C}+\frac{5F_6}{3C^2}\right)^2. \tag{126}$$

Obviously, when $k \to 1$, $(k' \to 0)$, the double periodic solutions (123) and (125) reduce to new types of the topological (kink-like) and nontopological multi-solitary wave solutions, respectively.

4.5. *Exact solutions of the FL model*

For cases D, H, I, J, K, L, M.1, M.2, N, and O, the constrained NKG equation has the form

$$\Box\phi = F_2\phi + F_3\phi^2 + F_4\phi^3 \tag{127}$$

$$(\tilde{\nabla}\phi)^2 = F_0 + F_2\phi^2 + \tfrac{2}{3}F_3\phi^3 + \tfrac{1}{2}F_4\phi^4. \tag{128}$$

The cases M.2, I, N and O correspond to $F_0 = 0$.

In [6] we also established the deformation relations between the constrained FL model and the constrained ϕ^4 system. Starting from every one solution of the ϕ^4 system, we can get a corresponding solution of (127). Here are three multi-solitary wave solutions:

$$\phi_{sol1} = \frac{1}{\pm\sqrt{F_2}\mathrm{e}^{\pm\sqrt{F_2}\xi} - C_2} - C_1 \tag{129}$$

with C_1, C_2 being determined by

$$C_1(F_3C_1 - F_4C_1^2 - F_2) = 0 \tag{130}$$

$$F_4 - \frac{5}{2}\left(\frac{F_3}{3} - C_1F_4\right)C_2 + \frac{1}{2}(3C_1^2F_4 - 2F_3C_1 + F_2)C_2^2 = 0 \tag{131}$$

$$\phi_{sol2} = \frac{-3F_2}{\text{sign } F_3\sqrt{(F_3^2 - \frac{1}{2}F_2F_4)}\,\text{ch}\,\sqrt{F_2}\xi - F_3} \tag{132}$$

and

$$\phi_{sol3} = \frac{-1}{\pm\sqrt{A}\,\text{ch}\,\sqrt{B}\xi + C_2} - C_1 \tag{133}$$

with

$$C_1 = \frac{F_3 \pm \sqrt{F_3^2 - 4F_2F_4}}{2F_4} \tag{134}$$

$$C_2 = \frac{-F_4(F_3 \pm 3\sqrt{F_3^2 - 4F_2F_4})}{3(F_3^2 - 4F_2F_4 \pm F_3\sqrt{F_3^2 - 4F_2F_4})} \tag{135}$$

$$B = \frac{1}{2F_4}(F_3^2 - 4F_2F_4 \pm F_3\sqrt{F_3^2 - 4F_2F_4}) \tag{136}$$

and

$$A = \frac{-F_4^2F_3(-F_3 \pm 3\sqrt{F_3^2 - 4F_2F_4})}{9(F_3^2 - 4F_2F_4 \pm F_3\sqrt{F_3^2 - 4F_2F_4})}. \tag{137}$$

4.6. Exact solutions of the shifted FL model

The following constrained NKG equation system:

$$\Box\phi = F_1 + F_2\phi + F_3\phi^2 + F_4\phi^3 \tag{138}$$

$$(\tilde{\nabla}\phi)^2 = 2F_1\phi + F_2\phi^2 + \tfrac{2}{3}F_3\phi^3 + \tfrac{1}{2}F_4\phi^4 \tag{139}$$

is related to cases E, F.3, G.3, M.3 and P of the last section ($F_3 = 0$ for cases F.3 and G.3).

It is quite easy to see that the constrained NKG system (138) and (139) can be solved by means of the FL model discussed in the last section. Using a (shift) transformation

$$\phi = g + c \tag{140}$$

with c being given by

$$F_1 + cF_2 + c^2F_3 + c^3F_4 = 0 \tag{141}$$

the equation system becomes

$$\Box g = g_2g + g_3g^2 + F_4g^3 \tag{142}$$

$$(\tilde{\nabla}\phi)^2 = g_0 + g_2g^2 + \tfrac{2}{3}g_3g^3 + \tfrac{1}{2}F_4g^4 \tag{143}$$

while g_i are given by

$$g_0 = c(2F_1 + cF_2 + \tfrac{2}{3}c^2F_3 + \tfrac{1}{2}c^3F_4) \tag{144}$$

$$g_2 = F_2 + 2cF_3 + 3c^2F_4 \tag{145}$$

$$g_3 = \tfrac{2}{3}F_3 + 2cF_4. \tag{146}$$

The exact solutions of (142) and (143) have been discussed in the last section.

4.7. Exact solutions of the ϕ^5 model

The related constrained NKG equations of the ϕ^5 model are

$$\Box\phi = F_2\phi + F_3\phi^2 + F_4\phi^3 + F_5\phi^4 \tag{147}$$

$$(\tilde{\nabla}\phi)^2 = F_0 + F_2\phi^2 + \tfrac{2}{3}F_3\phi^3 + \tfrac{1}{2}F_4\phi^4 + \tfrac{2}{5}F_5\phi^5. \tag{148}$$

To our knowledge, there is no exact known solution of (147) in the literature. Using the general discussions of [11] (or by direct calculation), a special type of the system (147) and (148) can be written as

$$\int^{\phi} \frac{\mathrm{d}y}{\sqrt{F_0 + F_2y^2 + \frac{2}{3}F_3y^3 + \frac{1}{2}F_4y^4 + \frac{2}{5}F_5y^5}} = \xi + \xi_0 \tag{149}$$

with ξ being given by (89). To write out some explicit solution of (149) is still quite difficult because of the difficulty of the integration. To get some explicit solitary wave solution of (147), one may use the nonstandard truncation approach of the Painlevé analysis given in [12]. Here we write down only one exact solution but omit the detailed derivation procedure and other possible solutions because of their complexity. Further details on the solutions of the ϕ will be reported in a separate paper [13]. In [13], we see that the ϕ^5 model is useful not only to get some special solutions of the NCSF but also to solve other nonlinear models, such as ϕ^8 models.

The ϕ^5 model (147) for $F_3 \neq 0$ possesses a multi-solitary wave solution

$$\phi = q + \frac{9q^4}{4p(3F_0 + 8F_3q^3)}\chi^2 \tag{150}$$

with p being an arbitrary constant and χ being given by ($c_1 = \exp(k_1\xi_0)$)

$$\frac{(\chi q^2 - A_-)^{B_-}(\chi q^2 - A_+)^{B_+}}{(\chi q^2 + A_-)^{B_-}(\chi q^2 + A_+)^{B_+}} = c_1\exp(k_1\xi) \tag{151}$$

where ξ is still determined by (89) and

$$A_\pm = \tfrac{2}{9}\sqrt{2}F_3qp\left(-8F_3q^3 \pm \sqrt{3F_0(32F_3q^3 + 15F_0)}\right) \tag{152}$$

$$B_\pm = \tfrac{4}{9}B_\pm\left(45F_0^2 \pm 8F_3q^3\sqrt{3F_0(32F_3q^3 + 15F_0)} + 96F_0F_3q^3\right) \tag{153}$$

$$k_1 = -\frac{4}{27q}\sqrt{-F_3(3F_0 + 8F_3q^3)pF_0q}(32F_3q^3 + 15F_0)(-15F_0 + 8F_3q^3) \tag{154}$$

$$q = -\frac{3q_1}{4F_3q_2} \tag{155}$$

$$\begin{aligned} q_1 = {} & -270F_5F_0^2F_4F_2^4 - 780F_5F_0^2F_2^3F_3^2 + 600F_5F_0^3F_4F_3^2F_2 + 108F_5F_0F_2^6 \\ & -400F_5F_0^3F_3^4 + 225F_3F_2^5F_4F_0 - 320F_3^3F_0^2F_2^2F_4 + 570F_3^3F_2^4F_0 \\ & -90F_3F_2^7 + 520F_3^5F_0^2F_2 \end{aligned}$$

$$\begin{aligned} q_2 = {} & 810F_5F_0^2F_2^3F_4 + 180F_5F_0^2F_2^2F_3^2 - 432F_5F_0F_2^5 - 360F_3^3F_0^2F_4F_2 - 675F_3F_0F_2^4F_4 \\ & +40F_3^3F_0F_2^3 + 360F_2^6F_3 - 120F_3^5F_0^2 \end{aligned}$$

if the parameters F_i satisfy the conditions

$$144F_5F_0q^5 + 9F_0^2 + 48F_0F_3q^3 + 64F_3^2q^6 = 0 \tag{156}$$

$$12F_2q^2 + 15F_0 + 16F_3q^3 = 0. \tag{157}$$

More concretely, if we take

$$F_0 = \tfrac{1}{2} + \tfrac{3}{10}\sqrt{5} \qquad F_3 = -\tfrac{16}{45}\sqrt{5} \qquad q = \tfrac{3}{8}\sqrt{5} \qquad p = 1$$

the χ function has a simple form

$$\frac{(\sqrt{10}\chi+1)^3(\sqrt{10}\chi-3)}{(\sqrt{10}\chi-1)^3(\sqrt{10}\chi+3)}=c_1\exp\left(\frac{12\sqrt{10}}{25}\xi\right). \tag{158}$$

4.8. On the exact solutions of the special ϕ^α model

In the last section, case R is related to a special ϕ^α model for arbitrary real α,

$$\Box\phi=F_2\phi+F_4\phi^3+F_\alpha\phi^{\alpha-1} \tag{159}$$

$$(\tilde{\nabla}\phi)^2=F_2\phi^2+\frac{1}{2}F_4\phi^4+\frac{2}{\alpha}F_\alpha\phi^\alpha. \tag{160}$$

One special type of solutions can be expressed by the general integration

$$\int^\phi\frac{\mathrm{d}y}{\sqrt{F_2y^2+\frac{1}{2}F_4y^4+\frac{2}{\alpha}F_\alpha y^\alpha}}=\xi+\xi_0 \tag{161}$$

with ξ being given by (89). In my knowledge, there is no known explicit function to express the integration of (161) for general real α except for $\alpha=3,5,6,8$. For $\alpha=3$, 5, and 6, the results have been discussed in the previous sections. In [13], we report the results for $\alpha=8$.

5. Summary and discussion

In summary, for the nonlinear coupled scalar field equations there are more abundant solitary wave solutions than for single scalar field models. For *every* selected single scalar field model (5), there may be some types of special solutions which are also solutions of the NCSF equations (1) and (2). After restricting the functions G and F in (5) and (6) as polynomial functions of ϕ up to ϕ^6, we have obtained 30 types of concrete exact solitary wave solutions and conoidal wave solutions of the NCSF by means of the ϕ^4 , ϕ^6, ϕ^3, ϕ^5 and $\phi^3+\phi^4$ (FL) models. If we do not put any constraint on the model parameters, there exists only one possible nontrivial polynomial selection (case A.2) for $N\leqslant 6$. Actually, we believe that case A.2 is the only polynomial selection for any N without any constraint on the model parameters and we have checked the conclusion by computer algebras up to $N=10$. Because of the difficulty in calculations, we cannot list all the possible polynomial selections here for $N>6$. Two special types of ϕ^α models for arbitrary real α can also be used to solve the NCSF.

From (1) and (2) we know that if we make the transformations

$$\phi\leftrightarrow\psi\qquad a_i\leftrightarrow b_i \tag{162}$$

the mode equations are form invariant. So we can get another 32 types of different exact solutions by using the transformations (162) to the solutions obtained in sections 3 and 4.

To understand the richness of the solitary wave in coupled nonlinear scalar fields, we can compare the travelling wave solution of the model with the classical mechanics. For the travelling wave solutions of a generalized nonlinear coupled scalar field system, we have

$$\phi_{TT}=-V_\phi(\phi,\psi)\equiv-\frac{\partial V}{\partial\phi} \tag{163}$$

$$\psi_{TT}=-V_\psi(\phi,\psi)\equiv-\frac{\partial V}{\partial\psi} \tag{164}$$

with

$$T=\frac{\sum_{i=1}^D k_ix_i+\omega t}{\sqrt{\sum_{i=1}^D k_i^2-\omega^2}}. \tag{165}$$

Comparing (163) and (164) with the classical mechanics, the equation system (163) and (164) describes a 'ball' (particle) that is rolling on a camber, $z = -V(x = \phi, y = \psi)$, without friction. A kink-like solitary wave is corresponding to the ball rolling from one peak of $-V$ to another peak with the same height while a 'bell' or 'ring' type solitary wave corresponds that the ball rolls down a peak and comes back to the same peak finally. For a single scalar field, the similar mechanical simulation is a particle moving in one-dimensional space ($x = \phi$) with potential $V(x)$. There may be various (or even infinitely many continuous) degenerate minima of V in two-dimensional 'space' $\{x = \phi, y = \psi\}$ and there may be various ways from one peak to another (or come back to the original peak). However, for a single scalar field, there exists only one way from one peak to another (or come back to the same) peak. That is why we can obtain more abundant solitary wave solutions of coupled scalar fields than those of single scalar fields.

In principle, one may obtain some special solutions of the NCSF system (1) and (2) for every given G by solving the model equations (5) and (6) with (7). Further details on the solutions of the system (5) and (6) with (7) will be discussed in future studies.

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